# Two Stage vs. One Stage Solution

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This problem simply uses momentum and one dimensional motion with a proof that an inequality is satisfied. Supposed that the mass which is ejected backwards is $-dm$ and the rocket goes up with some velocity $v$ and mass $M_{total}$. When the fuel is ejected, let the mass which is ejected backwards be $dm$ and at a speed which is $v-u$ relative to the rocket. Conservation of momentum gives us

$$M_{total}v = (m + dm)(v + dv) + (-dm)(v - u) \tag{1}$$

$$M_{total}dv = -udm \tag{2}$$

Notice that in (2) we have ignored the $dmdv$ term for the sake of simplification. We divided (2) by $M_{total}$ and integrate from initial velocity to final velocity.

$$\int_0^{v_f} dv = -\int_{M_i}^{M_f} \frac{u}{M_{total}} dm \tag{3}$$

We know that our initial velocity was 0 and that our initial mass for the first rocket is $M + 2m$ and our final mass is $M$. We get

$$v_f = u \ln\left(\frac{M+2m}{M}\right) \Rightarrow d_{max} = \frac{u^2\left(\ln\left(\frac{M+2m}{M}\right)\right)^2}{2g} \tag{4}$$

the last result is a simple result of 1-d motion. We will do this process again for rocket B. Remember that the start of the second burn is when the upwards velocity from the first burn is exactly zero. We essentially have the process above for the two stages.

$$d_{max} = \frac{u^2\left(\ln\left(\frac{M+2m}{M+m}\right)\right)^2}{2g} + \frac{u^2\left(\ln\left(\frac{M+m}{M}\right)\right)^2}{2g} \tag{5}$$

Now we need to find which is larger. To do this, we should start with a hypothesis of the inequality and see where it takes us. Assume that the two stage rocket goes higher

$$\frac{u^2\left(\ln\left(\frac{M+2m}{M+m}\right)\right)^2}{2g} + \frac{u^2\left(\ln\left(\frac{M+m}{M}\right)\right)^2}{2g} > \frac{u^2\left(\ln\left(\frac{M+2m}{M}\right)\right)^2}{2g} \tag{6}$$

Eliminating constants and using log properties we will simplify (6).

$$\ln(M+2m)\ln(m+M) + \ln(m+M)\ln(M) - \ln(M+m)^2 < \ln(M+2m)\ln(M) \tag{7}$$

Upon careful realization (7) can actually be factored into

$$(\ln(M+2m) - \ln(M+m))(\ln(M+n) - \ln(M)) < 0 \tag{8}$$

Notice that since $M, m > 0$ that each factor is positive and so we arise at a contradiction and we must reverse the sign of our prediction so the single stage rocket goes higher.