# Comet and Satellite Intersection Solution

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Note that the energy of the orbit is zero because the eccentricity of a parabola is 1 and the eccentricity definition of a parabolic orbit dictates that the energy is zero. Start off by calling the dotted line to have distance $q$. With the fact that the $\dot{r}$=0 at a distance of $q$ we can solve for $L$.

$$0 = \frac{L^2}{2mq^2} - \frac{GMm_s}{q} \quad \Rightarrow \quad L = \sqrt{2GMm_s^2 q} \tag{1}$$

Here we used the gravitational potential and $M$ is the mass of the star, while $m_s$ is the mass of the satellite. The value of $\dot{r}$ can be solved for with

$$\frac{dr}{dt} = \sqrt{\frac{2}{m_s}\left(\frac{GMm_s}{r} - \frac{L^2}{2mr^2}\right)} \quad \Rightarrow \quad t = \int_q^{r_{max}} \frac{dr}{\sqrt{\frac{2}{m_s}\left(\frac{GMm_s}{r} - \frac{L^2}{2mr^2}\right)}} \tag{2}$$

Substitute $L$ which we got from equation (1) into (2) and simplify the integral to

$$2\sqrt{\frac{m}{2GMm_s}} \int_q^{r_{max}} \frac{rdr}{\sqrt{r-q}} \tag{3}$$

The last integral we will solve with a substitution of $u = r - q$ and plugging it back gives

$$\int_q^{r_{max}} \frac{rdr}{\sqrt{r-q}} = \frac{2(r_{max}-q)^{3/2}}{3} + 2q\sqrt{r_{max}-q} \quad \Rightarrow \quad \frac{2\sqrt{2}}{3}\sqrt{\frac{r_{max}-q}{GM}}\,(r_{max}+2q) = t \tag{4}$$

$$d = \frac{2\sqrt{2}}{3}\sqrt{\frac{r_{max}-q}{GM}}\,(r_{max}+2q) \quad \Rightarrow \quad d = \frac{2\sqrt{2}}{3}\sqrt{\frac{r_{max}^3}{GM}}\left(1 + \frac{2q}{r}\right)\sqrt{1 - \frac{q}{r}} \tag{5}$$

The key to notice here is that the quantity $\sqrt{\frac{r_{max}^3}{GM}}$ is actually a constant because of the way the problem was asked, we can treat the path in which the satellite traces within the comets trajectory to be essentially as if the satellite was traveling through an elliptical orbit. In this case, we only needed to maximize the quantity of $\frac{q}{r}$.

$$f\left(\frac{q}{r}\right) = \left(1 + \frac{2q}{r}\right)\sqrt{1 - \frac{q}{r}} \tag{6}$$

$$\frac{df}{dt} = 0 = \frac{3 - 6\frac{q}{r}}{2\sqrt{1 - \frac{q}{r}}} \tag{7}$$

Equation (7) is satisfied for $\frac{q}{r}$ is $\frac{1}{2}$ and plugging that into (6) we find that the max is $\sqrt{2}$ and so putting it all together by subbing in the max of the function $f$ into (5)

$$\frac{3d}{2} = \sqrt{\frac{r_{max}^3}{GM}} \quad \Rightarrow \quad \boxed{r_{max} = \sqrt[3]{\frac{9d^2GM}{4}}} \tag{8}$$