Projectile Motion Solutions

M. Yu

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1. $\boxed{\dfrac{2d}{\sqrt{\frac{2h}{g}}}}$ The ball travels a total distance of $2d$ while trapped between the walls and falls a distance of h in a time if $\sqrt{\frac{2h}{g}}$. Thus the initial velocity is the distance divided by time.

2. $\boxed{9}$ There is a simple symmetry argument that can be made where instead of tracing the bounces, one can simply allow the ball to go through the table into another table. Since the ball is directed at a 45 degree angle, if the ball covers 73 feet in the vertical direction, then it necessarily covers 73 feet in the horizontal direction as well. In order for the ball to arrive at a hole we must have the following be true:

$$73k + 63 \equiv 0 \pmod{120}$$

We can solve this equation by using the idea of an inverse in a mod.

$$\begin{aligned} 73k &\equiv -63 \pmod{120} \\ k &\equiv -63 \cdot 97 \pmod{120} \\ k &\equiv 9 \pmod{120} \end{aligned}$$

3. $\boxed{v_2 = 2v\cos\alpha\sin\theta(\tan\alpha - \tan\theta)}$ The slope of the plane is $\tan\theta$ and thus the ball hits the plane with the coordinates satisfy $\frac{y}{x} = \tan\theta$.

$$h = v\sin\alpha t - \frac{1}{2}gt^2, \qquad \frac{h}{\tan\theta} = v\cos\alpha t \tag{1}$$

$$\tan\theta = \frac{v\sin\alpha t - \frac{1}{2}gt^2}{v\cos\alpha t} \quad \Rightarrow t = \frac{2v\cos\alpha(\tan\alpha - \tan\theta)}{g} \tag{2}$$

Equations (1) and (2) deal with the motion of the watermelon. Consider the time at which the cookies reach Matthew, it is equal to the time of flight of the watermelon. The acceleration a long the plane is $g\sin\theta$ and the distance the cookies travel is $\frac{h}{\sin\theta}$

$$\frac{v_2}{g\sin\theta} = \frac{2v\cos\alpha(\tan\alpha - \tan\theta)}{g} \tag{3}$$

$$v_2 = 2v\cos\alpha\sin\theta(\tan\alpha - \tan\theta) \tag{4}$$

4. $\boxed{v_{min} = \sqrt{2gR(1+\sqrt{2})}}$ Patrick's motion through the air will take a parabolic trajectory. If Patrick is to minimize the velocity of take off, there will be two points on the boulder which his path will be tangential to. Call the two points A and A' Let those two points make an angle ϕ with the horizontal line connecting them. Consider the center of the boulder O and the points A and A'. It is obvious that A and A' are above the center of the circle. The midpoint of line segment AA' is M and $\angle MOA = \phi$. From point A consider the velocity to be v_2

$$v_2\sin\phi t + \frac{1}{2}gt^2 = 0, \quad v_2\cos\phi t = 2R\sin\phi \tag{5}$$

In the above equation, t is the time it takes for the Patrick to fly from A to A', and R is the radius of the boulder (d/2). Solve for v_2 by eliminating t we get

$$v_2^2 = \frac{gR}{\cos\phi} \tag{6}$$

The next equations come from conservation of energy, v is the original take off velocity. After which, substitute in the value of v_2^2.

$$\frac{1}{2}mv^2 = \frac{1}{2}mv_2^2 + mg(R + R\cos\phi) \tag{7}$$

$$\frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{gR}{\cos\phi}\right) + mgR(1+\cos\phi) \tag{8}$$

$$v^2 = 2gR\left(\frac{1}{2\cos\phi} + 1 + \cos\phi)\right) \tag{9}$$

We want to minimize this equation. One could take he derivative, or in the spirit of being slick apply the handy AM-GM equation to minimize the terms with $\cos\phi$. (As an exercise, verify the validity of AM-GM for n terms with Jensen's Inequality. Avoid the dark arts of Lagrange multipliers)

$$\frac{\frac{1}{2\cos\phi} + \cos\phi}{2} \geq \sqrt{\cos\phi\frac{1}{2\cos\phi}} = \frac{\sqrt{2}}{2} \quad \Rightarrow \frac{1}{2\cos\phi} + \cos\phi \geq \sqrt{2} \tag{10}$$

Putting it all together, our minimum velocity is

$$v_{min} = \sqrt{2gR(1+\sqrt{2})} \tag{11}$$

5. $\boxed{\frac{2\ell}{3v}}$ We have an equilateral triangle so consider breaking up Patrick's velocity into components so that one component points to the center of the triangle. The component of velocity towards the center of the triangle is $v\sin 30$. The circumradius of a equilateral triangle is $\frac{\sqrt{3}\ell}{3}$ in length and thus time is just distance over velocity.