Feynman Disk Solution

M. Yu

January 2015

1 Introduction

Feynman stumbled on this problem at a cafeteria in Cornell when he was fooling around with throwing a spinning disk into the air. He found that the ratio of wobbling to rotation is 2 : 1 for some angle the plate is tilted in during its motion. He later went on to ponder how the electron orbits start to move in relativity and formulating quantum electrodynamics which he eventually won a Nobel Prize for; this problem is a big deal and while it is not original, it is worth doing a good derivation of.

2 Solution

The motion of the rigid body in the inertial frame can be expressed in terms of the Euler angles θ, ϕ, ψ. Consider the problem in the frame of the disk. Let the frame of standing on the disk be D' and the inertial frame be D. In the figure below the axis all correspond to the ones which where given, it is simply a different point of view which is intended to make equations easier to follow.

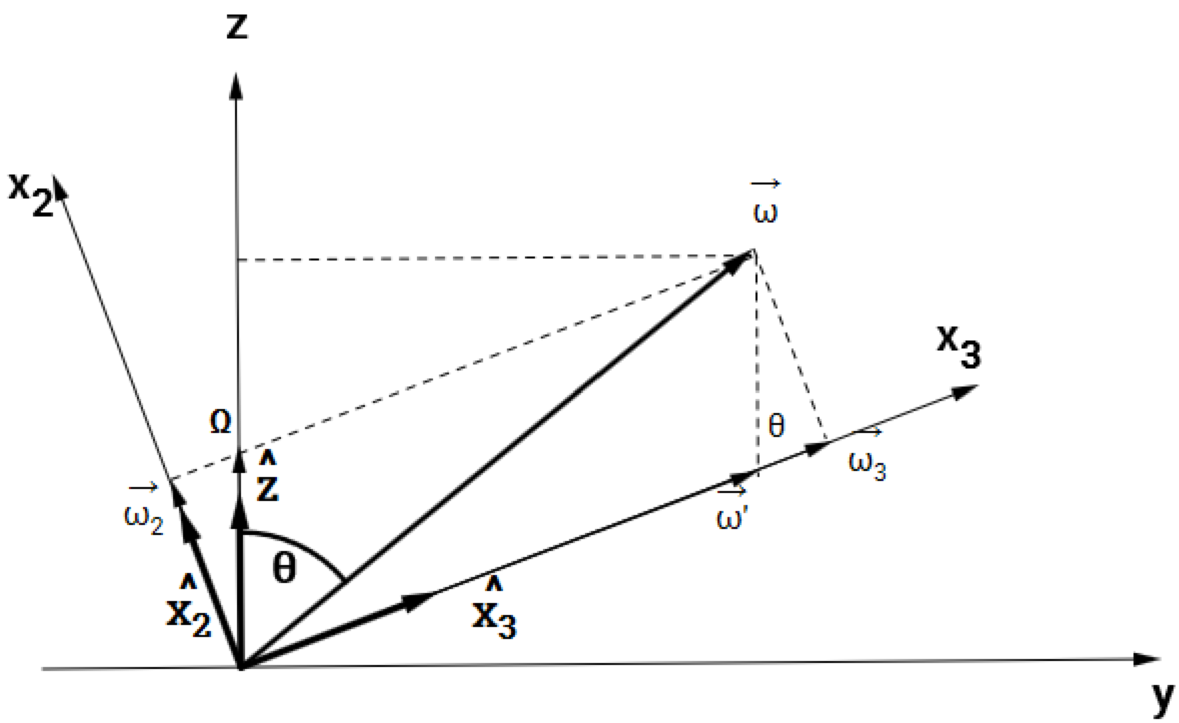
Notice first that ω_2 and ω_3 are in the frame of the disk that is shaded and $\dot{\theta} = 0$ both of which should make sense the latter being that it spins at a constant rate. Start off now with the angular velocity which is generally $\omega = \omega_1 \hat{\mathbf{x_1}} + \omega_2 \hat{\mathbf{x_2}} + \omega_2 \hat{\mathbf{x_3}}$. We have that $\dot{\Psi}\hat{\mathbf{x_3}}$is the angular velocity of the body in the frame of D', to shift it to the lab frame we need to add on some extra terms particularly $\dot{\phi}\hat{\mathbf{z}} + \dot{\theta}\hat{\mathbf{x_1}}$ in order to shift ω to the lab frame D.

$$\omega = \dot{\psi}\hat{\mathbf{x_3}} + (\dot{\phi}\hat{\mathbf{z}} + \dot{\theta}\hat{\mathbf{x_1}}) \tag{1}$$

Rewrite $\hat{\mathbf{z}}$ in terms of the $\hat{\mathbf{x_1}}$ and $\hat{\mathbf{x_2}}$ basis vectors $\hat{\mathbf{z}} = \cos\theta\hat{\mathbf{x_3}} + \sin\theta\hat{\mathbf{x_2}}$. Combine equations (10) and (11) to obtain

$$\omega = (\dot{\psi} + \dot{\phi}\cos\theta)\hat{\mathbf{x_3}} + \dot{\phi}\sin\theta\hat{\mathbf{x_2}} + \dot{\theta}\hat{\mathbf{x_1}} \tag{2}$$

The reader is encouraged to look at the diagram provided in the problem to make sense of these equations. What will follow will utilize these equations in an equivalent mannor that is perhaps more easily understood and will help to solve the problem



Consider the diagram above and obtain the following for $\hat{\omega}$.

$$\omega = \omega' \hat{\mathbf{x_3}} + \Omega \hat{\mathbf{z}} \tag{3}$$

$$\omega = \omega_2 \hat{\mathbf{x_2}} + \omega_3 \hat{\mathbf{x_3}} \tag{4}$$

We are primarily interested in $\dot{\phi}$ as that gives the precession velocity which will cause the wobbling as well as the ω' term on the $\hat{\mathbf{x_3}}$ vector. By matching terms from equation (12) with equation (10) we easily see that $\omega' = \dot{\mathbf{\Psi}}$ and $\Omega = \dot{\phi}$. We also have by comparing to equation (13) that

$$\omega_3 = \dot{\psi} + \dot{\phi} \cos\theta = \omega' + \Omega \cos\theta \tag{5}$$

$$\omega_2 = \dot{\phi} \sin\theta = \Omega \sin\theta \tag{6}$$

Remember how we established Ω. Notice that the moment of inertia I_3 about $\hat{\mathbf{x_3}}$ is $\frac{mr^2}{2}$ while the moments I about $\hat{\mathbf{x_2}} = \hat{\mathbf{x_1}} = \frac{mr^2}{4}$ due to symmetry. The reader is also encouraged as an exercise to diagonalize the inertial tensor to verify the moments about the principal axises. One might also find a surprising result when considering the Euler equations for moments of inertial which are not at all symmetric but that is a topic for another discussion.

2.1 Defense Against the Dark Arts

It is time to turn equations of velocity into equations of position and I say we avoid the dark arts (otherwise known as the Lagrangian method) and approach with a more physical method of torques. It was said that Feynman did his problem with the Lagrangian, but the same answer will arise regardless. Recall that $\tau = \frac{d\mathbf{L}}{dt}$, lets define a variable $\dot{\lambda} = \dot{\psi} + \dot{\phi}\cos\theta$ this is just to make taking derivatives a bit less tedious. Simply multiply each angular velocity component with its respective moment of inertia and we get

$$\mathbf{L} = I_3 \dot{\lambda} \hat{\mathbf{x_3}} + I \dot{\phi} \sin\theta \hat{\mathbf{x_2}} + I \dot{\theta} \hat{\mathbf{x_1}} \tag{7}$$

Similarly to how the Euler equations were derived, the unit vectors $\hat{\mathbf{x_2}}$ and $\hat{\mathbf{x_3}}$ both change with time so we proceed by taking derivatives of (16) copiously applying the product rule

$$\frac{d\mathbf{L}}{dt} = I_3 \frac{d\dot{\lambda}}{dt} \hat{\mathbf{x_3}} + I \frac{d(\dot{\phi \sin\theta})}{dt} \hat{\mathbf{x_2}} + I \frac{d\dot{\theta}}{dt} \hat{\mathbf{x_1}} + \left(I_3 \dot{\lambda} \frac{d\hat{\mathbf{x_3}}}{dt} + I \dot{\phi} \sin\theta \frac{d\hat{\mathbf{x_2}}}{dt} + I \dot{\theta} \frac{d\hat{\mathbf{x_1}}}{dt} \right) \tag{8}$$

Break down each of the basis vector derivatives so that they are in terms of the vectors themselves and simplify the resulting equation

$$\frac{d\hat{\mathbf{x_3}}}{dt} = -\dot{\theta}\hat{\mathbf{x_2}} + \dot{\phi}\sin\theta\hat{\mathbf{x_1}} \tag{9}$$

$$\frac{d\hat{\mathbf{x_2}}}{dt} = \dot{\theta}\hat{\mathbf{x_3}} - \dot{\phi}\cos\theta\hat{\mathbf{x_1}} \tag{10}$$

$$\frac{d\hat{\mathbf{x_1}}}{dt} = -\dot{\phi}\sin\theta\hat{\mathbf{x_3}} + \dot{\phi}\cos\theta\hat{\mathbf{x_2}} \tag{11}$$

To get a grasp of what these equations are saying, take the first of the three. Refer back to the figure on the problem statement and the equation says that for some change in theta, $\hat{\mathbf{x_3}}$ moves some distance in the $\hat{\mathbf{x_2}}$ direction. Try to show that a change in ϕ causes $\hat{\mathbf{x_3}}$ to change in the $\hat{\mathbf{x_1}}$ direction and similarly for every other equation. Cleaning equation (17) up by subbing in all the variables we get

$$\frac{d\mathbf{L}}{dt} = I_3\ddot{\lambda}\hat{\mathbf{x_3}} + \left(I\ddot{\phi}\sin\theta + 2I\dot{\theta}\dot{\phi}\cos\theta - I_3\dot{\lambda}\dot{\theta}\right)\hat{\mathbf{x_2}} + \left(I\ddot{\theta} - I\dot{\phi}^2\sin\theta\cos\theta + I_3\dot{\lambda}\dot{\phi}\sin\theta\right)\hat{\mathbf{x_1}} \tag{12}$$

Torque is provided by gravity is $mgl\sin\theta$ in the $\hat{\mathbf{x_1}}$ direction. We can thus set each of the other components besides $\hat{\mathbf{x_1}}$ equal to zero and $\hat{\mathbf{x_1}}$ equal to $\hat{\mathbf{x_1}}$.

$$\ddot{\lambda} = 0 \ \Rightarrow \dot{\lambda} = constant = \omega_3 \tag{13}$$

$$I\ddot{\phi}\sin\theta + \dot{\theta}\left(2I\dot{\phi}\cos\theta - I_3\dot{\lambda}\right) = 0 \tag{14}$$

$$\left(mgl + I\dot{\phi}^2\cos\theta - I_3\omega_3\dot{\phi}\right)\sin\theta = I\ddot{\theta} \tag{15}$$

In the case when the disk exhibits uniform circular motion we have $\dot{\theta} = 0$. Equation (24) becomes a quadratic which we can solve!

$$I\dot{\phi}^2\cos\theta - I_3\omega_3\dot{\phi} + mgl = 0 \tag{16}$$

$$\dot{\phi} \equiv \Omega = \frac{I_3\omega_3}{2I\cos\theta}\left(1 \pm \sqrt{1 - \frac{4mglI\cos\theta}{I^2\omega_3^2}}\right) \quad \Rightarrow \Omega = \frac{I_3\omega_3}{I\cos\theta} \tag{17}$$

we are going with the approximation that the value under the square root is insignificant. We know that rotation is simply ω_3. this looks really good, don't forget about the moments of inertias given. Putting it all together we find that

$$\frac{\Omega}{\dot{\psi}} = \boxed{2} \tag{18}$$

and we are done. Phew!