Magnetism and Induction

M. Yu

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1 Intro

Ampere's current law states that

$$\oint_c \mathbf{B} \cdot d\ell = \mu_0 I_{enc}.$$

Notice how convenient this is to do when the Amperian loop has high symmetry, as the line integral is easy to compute. This is basically just Gauss' law for currents and magnetic fields. More generally, we write the current using the current density $\mathbf{J}(x, y, z)$ so that

$$\oint_C \mathbf{B} \cdot d\ell = \mu_0 \oint_S \mathbf{J} \cdot d\mathbf{a} = \oint_S \nabla \times \mathbf{B} \cdot d\mathbf{a}.$$

The last equality stems from Stokes' theorem and in this regard $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$. We furthermore know that $\nabla \cdot B = 0$ stating that the divergence of the magnetic field is zero (proof?). These two conditions on the $\mathbf{B}$ field make it so that it is unique. We recover Ampere's law from Biot-Savart in the following way. We can replace the current part of Biot-Savart with the current density and integrate over the volume in question

$$\mathbf{B} = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}(\mathbf{r'}) \times \hat{r}}{r^2} d\tau. \tag{1}$$

The $\mathbf{r'}$ is for the coordinate of the volume, while $\mathbf{r}$ is the vector from a volume element to a point in question. It is left to the reader to take the curl and divergence of (1) to verify the results about the magnetic field as discussed earlier. This requires quite the gymnastics performance in vector calculus. The final formula which we talk about is the relationship between the electromotive force and magnetic flux Φ

$$\mathscr{E} = -\frac{d\Phi}{dt}. \tag{2}$$

Remember that the electromotive force is an integral of force per unit charge.

2 Problems

1. Given a magnetic field pointing in the z direction, $B_z = B_0(1 + z)$. A hoop of radius r, mass m and resistance R falls through the field. What is the terminal velocity of the hoop?

2. Easy question, find the magnetic field at a distance d from a straight infinite wire carrying current I. Next, consider a quarter of a ring which has radius r_1, and another concentric quarter of a ring of r_2 such that $r_2 < r_1$. the ends of the rings are joined by vertical and horizontal pieces of wire. Find the magnetic field if a current I goes around the loop clockwise.