Lagrangian and Hamiltonian Mechanics Notes and Problems

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1 Review: Lagrangian Mechanics

Lagrangian Mechanics all started with Fermat. Fermat noticed that light traveled along paths that made the time either a maximum, minimum, or "saddle point" (or in general, an extremum) with respect to changes in the path. This eventually led to the observation that classical mechanics behaves in a similar way, known as Hamilton's principle.

Theorem 1.1 (Hamilton's Principle)**.** *The action integral of a physical system is stationary under variations of the path of the actual path*

Remember we defined the action integral as follows

$$S[q] = \int_{t_0}^{t_1} L(q(t), \dot{q}(t), t)dt \tag{1}$$

where $\dot{q} = \dfrac{dq}{dt}$ and L is a function that we know. Being stationary means that

$$\delta S = 0 = \left[S[q + \epsilon\eta(t)] - S[q] \right]\Bigg|_{\text{to linear order}} \tag{2}$$

After doing some math, we arrive at the following equation for determining q

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0 \tag{3}$$

I'll assert that this equation also works if we have more than one coordinate, in which case we just apply it to each q. So if we have $L = L(q_1, \dot{q}_1, \cdots, q_n, \dot{q}_n, t)$ then we would have n equations, namely

$$\frac{\partial L}{\partial q_1} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_1}}\right) = 0 \tag{4}$$

$$\vdots$$

$$\frac{\partial L}{\partial q_n} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_n}}\right) = 0 \tag{5}$$

That is all that we need for now, but Lagrangians are very general and a lot more theory can be developed. Such development is left as an exercise to the reader.

2 Hamiltonian Mechanics

2.1 Generalized Momenta

Definition 2.1. We define the generalized momenta as

$$p_j = \frac{\partial L}{\partial \dot{q}_j} \tag{6}$$

Remark. This can also be known as the *conjugate* or *canonical* momentum

Let's rewrite the Euler-Lagrange Equation using the generalized momentum.

$$\dot{p}_j = \frac{\partial L}{\partial q_j} \tag{7}$$

Notice, if L is independent of q_j then p_j is a conserved quantity.

Problem 2.1.1 Find the generalized momentum for a free particle of mass m. Does the generalized momentum remind you of something you already know about? Is it a conserved quantity? *Note*: A free particle is one that has no potential energy. Also $L =$ KE - PE, or in this case, $L =$ KE

2.2 The Hamiltonian

Definition 2.2. We now define the Hamiltonian as

$$\mathcal{H}(q, p, t) = \left[\sum_{j=1}^{n} p_j \dot{q}_j\right] - L(q, \dot{q}, t) \tag{8}$$

Remark. This is known as a *Legendre Transform.* It is also useful in other areas of physics (such as Thermodynamics) and for sounding smart.

What happens if we calculate the total differential? Well we have two ways to calculate it, the multivariate chain rule and the definition. Using the chain rule, we get

$$d\mathcal{H} = \left[\sum_{j=1}^{n} \frac{\partial \mathcal{H}}{\partial p_j} dp_j + \frac{\partial \mathcal{H}}{\partial q_j} dq_j\right] + \frac{\partial \mathcal{H}}{\partial t} \tag{9}$$

Using the definition of $\mathcal{H}$, we get

$$d\mathcal{H} = \left[\sum_{j=1}^{n} \dot{q}_j dp_j + p_j d\dot{q}_j\right] - \left[\sum_{j=1}^{n} \frac{\partial L}{\partial q_j} dq_j + \frac{\partial L}{\partial \dot{q}_j} d\dot{q}_j\right] - \frac{\partial L}{\partial t} dt \tag{10}$$

But we can simplify this using (6) and (7) to arrive at

$$d\mathcal{H} = \left[\sum_{j=1}^{n} \dot{q}_j dp_j - \dot{p}_j dq_j\right] - \frac{\partial L}{\partial t} dt \tag{11}$$

Since (9) and (11) must be equal, we get the following equations, known as *Hamilton's equations*

$$\frac{\partial \mathcal{H}}{\partial p_j} = \dot{q}_j \tag{12}$$

$$\frac{\partial \mathcal{H}}{\partial q_j} = -\dot{p}_j \tag{13}$$

$$\frac{\partial \mathcal{H}}{\partial t} = -\frac{\partial L}{\partial t} \tag{14}$$

Wow! Such a big discovery! This is such a new advancement. Just kidding, all we have done is rewrite the Euler-Lagrange Equations. Instead of having n second order differential equations, we have $2n$ first order differential equations.

Problem 2.2.1 Find the Hamiltonian for the harmonic oscillator for a particle of mass m and spring constant k using the definition above. Does the Hamiltonian remind you of something you already know about? *Note*: $PE = \frac{1}{2}kx^2$ and $KE = \frac{p^2}{2m}$. Again, $L =$ KE - PE

Hey, the Hamiltonian is just the energy isn't it? Not quite. It is only the energy if

1. Lagrangian is $L = L_0(q,t) + L_1(q,t)\dot{q}_i + L_2(q,t)\dot{q}_j\dot{q}_k$
2. Constraints are time-independent, so $T = L_2(q,t)\dot{q}_j\dot{q}_k$
3. Forces are conservative, so $V = -L_0(q)$

This is going to be true for most cases in mechanics, so it is safe to assume that $\mathcal{H} = T + V$. Now for some problems.

Problem 2.2.2 You already found the Hamiltonian for the harmonic oscillator, now what are the equations of motion? Do they agree with Lagrangian and Newtonian Mechanics?

Problem 2.2.3 Find the equations of motion of a particle under a central force. So $V = V(r)$. Note that $T \neq \frac{1}{2}m\dot{r}^2$, but rather $T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$

Problem 2.2.4 Use the Lagrangian $L = \frac{1}{2}m\dot{x}_j^2 - q\phi + qA_j\dot{x}_j$ for the EM field to determine $\mathcal{H}$

Remark. This is an example of where $L \neq T - V$, because of the last term in L, but it turns out that $\mathcal{H}$ ends up to be the total energy.

Problem 2.2.4 Super Ultra Bonus Challenge Use (12) and (13) to rederive Lorentz's force law $\frac{d}{dt}(m\dot{x}_j + qA_j) = q\dot{x}_j\frac{A_i}{x_j} - q\frac{\partial\phi}{\partial x_j}$ Note: In this problem, I put an implied sum over j, so anything with an index of j is being summed over 3 times (the index in this case refers to what dimension it refers to, so $x_1 = x$, $x_2 = y$, $x_3 = z$. Additionally, note the A_i in the partial on the right. You have to be very careful about indices.

Problem 2.2.4 Super Ultra Bonus Challenge with a Cherry on Top Use the definitions of $\mathbf{E} = -\nabla\phi - \frac{\partial\mathbf{A}}{\partial t}$ and $\mathbf{B} = \nabla \times \mathbf{A}$ to derive the usual form of the Lorentz force law, $\frac{d}{dt}\big(mv_j\big) = qE_j + q\big(\mathbf{v} \times \mathbf{B}\big)_j$ If you do this and show it to me, you will win all the brownie points, and you just might be not half bad at physics.