Gravity Problem Set

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1 Recapitulation of the Sermon

Polar Coordinates:

$$\mathbf{r} = r\mathbf{e_r} \tag{1}$$

$$\dot{\mathbf{r}} = \dot{r}\mathbf{e_r} + r\dot{\theta}\mathbf{e}_\theta \tag{2}$$

$$\ddot{\mathbf{r}} = (\ddot{r} - r\dot{\theta}^2)\mathbf{e_r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\mathbf{e}_\theta \tag{3}$$

Binet Equation:

$$\frac{d^2u}{d\theta^2} + u = -\frac{1}{ml^2u^2}f(\frac{1}{u}) \tag{4}$$

Where $l = r^2\dot{\theta} = \frac{L}{m}$ and $u = \frac{1}{r}$

2

Using Lagrangians, prove that the equation of motion for the radius r of a particle of mass m moving in a central potential V(r) is:

$$m\frac{d^2r}{dt^2} - mr\omega^2 = -\frac{dV}{dr} = F \tag{5}$$

3

Using conservation of energy and conservation of angular momentum, derive the following equation, known as the vis-viva equation:

$$v^2 = GM(\frac{2}{r} - \frac{1}{a}) \tag{6}$$

Where v and r are, respectively, the velocity and radius at any point in an elliptical orbit, and a is the semi-major axis.

4

The most efficient way for a spacecraft to get from Earth to Mars is through a slick maneuver called a Hohmann transfer (diagram on the next page). Essentially, our spacecraft will transfer from a circular orbit around the sun which passes very close to the Earth to a much larger circular orbit around the sun which passes very close to Mars. For this to happen, we will first boost our near-Earth circular orbit into an elliptical orbit which touches both the near-Earth circular and near-Mars circular orbits. We will "ride" that ellipse to Mars, and then do another velocity boost to shift from the elliptical orbit to the near-Mars circular orbit. Assume that Earth and Mars are in circular orbits around the sun with respective radii R_E and R_M. Let $R_M = \alpha R_E$, where α is some constant. Let the mass of the sun be M.

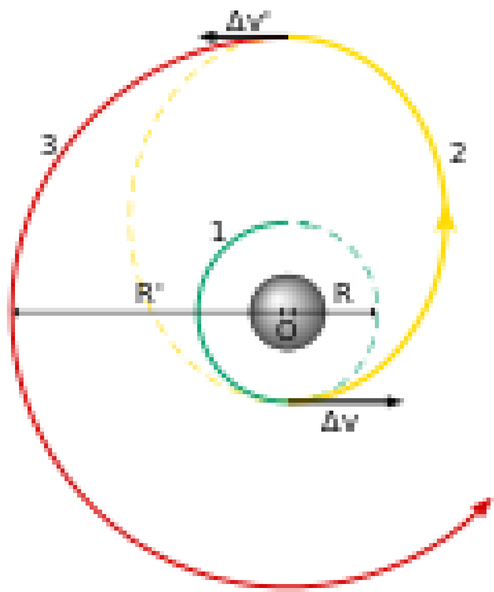

Figure 1: 1 represents the near-Earth circular orbit, 2 represents the elliptical orbit, and 3 represents the near-Mars circular orbit. The sun is at the center.

Using the vis-viva equation and your knowledge about circular orbits, calculate the magnitudes of the two necessary velocity boosts.

5

A particle moving in a central field has the spiral orbit

$$r = r_0 e^{k\theta} \tag{7}$$

Using the Binet equation and conservation of angular momentum, prove that the force law is inverse cube and that θ varies logarithmically with t.