Lagrangian Problems Part 1

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Solve the following problems without the standard method of Newtonian mechanics. The Lagrangian is defined to be $\mathscr{L} = T - U$ (kinetic - potential energy), the Euler-Lagrange equation is:

$$\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{q}}\right) - \frac{\partial \mathscr{L}}{\partial q} = 0.$$

1. Give the equation of acceleration of Patrick on a sled on a frictionless ramp inclined at an angle of θ using the Lagrangian. Prove that it agrees with $g\sin(\theta)$.

2. We had this problem in the gravitation pset about Patrick and his breakfast. Patrick has mass m and his breakfast has mass M. They initially start infinitely far away. When they are a distance r from each other find how fast Patrick and his breakfast are moving.

3. Matthew has Patrick tied onto the end of a string, forming a pendulum. Let the angle that the Patrick pendulum makes with the vertical be θ and the string have length ℓ. Find the equation of motion of the pendulum.

4. Take the first problem again. This time, the ramp has mass M and Patrick has mass m. When Patrick slides down the ramp, he also pushes the ramp backwards. Assume all contact surfaces are frictionless. Find the acceleration of the ramp.

   Remark. *Consider setting up two coordinate systems. Let the coordinate for the ramp be relative to a point far away from the ramp. You may set up the coordinate for Patrick the same way as problem 1.*

5. Here is a prelude to what is to come. A function $F(q_j, \dot{q}_j, t)$ which has its total time derivative zero is called a constant of motion, better known as a conserved quantity. As an example we will define the Hamiltonian:

   $$H = \sum_j q_j \frac{\partial \mathscr{L}}{\partial \dot{q}_j} - \mathscr{L}$$

   - Prove that the Hamiltonian, or total energy, is conserved given that the E-L holds by taking $\frac{dH}{dt}$.
   - Let $p_j = \frac{\partial \mathscr{L}}{\partial \dot{q}_j}$, we call this the generalized momentum. Show that if $\frac{\partial \mathscr{L}}{\partial q} = 0$, then momentum is a conserved quantity. (Use the E-L equation given above)

6. ***Challenge*:** Suppose you are ambitious enough to try and describe everything in the observable universe in one mathematical equation

   $$\mathscr{L} = \sqrt{g}\left(R - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \psi \not{D} \bar{\psi}\right).$$

   Learn all you can about this equation, come back, and teach me something you learned! You might want to search"Standard Model"