Special Relativity Guided Activity

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1 Postulates of Special Relativity

- The laws of physics are the same in all inertial (non-accelerating) frames of reference.
- The speed of light in free space is the same in all reference frames.

2 Lorentz Transformations

Now we will turn to determining how space and time coordinates transform between reference frames - and they do so through the Lorentz transformation. There are many roundabout ways to derive the Lorentz transform, but here we will guide you through an approach close to the OG.

F represents an inertial reference frame, which parametrizes space-time with two coordinates; t and x (as mentioned before, we are only concerning ourselves with 2 dimensions). From now on, F(t,x) represents space-time as viewed in frame F at time t and point x.

Now, despite the fact that time and space get distorted between reference frames, we can decide on a point at which all reference frames agree on the time and position. To put it more formally, we can normalize all reference frames with an event that occurs at the origin in all reference frames - let us call this event O. Then, F(O) = (0,0).

Now, let us define the operation of the Lorentz transform, which is a function of velocity v. What we want the Lorentz transform to do is establish a link between a stationary frame F and one which is moving at a constant v relative to F; let us call the latter F_v. We will call the Loretntz transform operator L_v, so that $F_v = L_v(F)$. Technically, the Lorentz transform operates on frames, but for shorthand I will write

$L_v(t, x)$. From our Normalization, we know that $L_v(0,0) = (0,0)$.

Now, let us turn to constructing L_v. We know that we want L_v to transform a 2x1 matrix into another 2x1 matrix:

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = L_v \begin{pmatrix} t \\ x \end{pmatrix}$$

I will assert that L_v is a linear operator. Here is a brief explanation: if L_v were not linear, it would take an inertial frame of reference and make it non-inertial, which is not what we want. Since L_v is a linear transformation between $R \times R$ and $R \times R$, we can represent it as a matrix. And that matrix is a function of v. And when I say function, you say Taylor series. So we are going to construct a Taylor series of a matrix function of v centered around $v = 0$. So here we go

Let us begin with a first order approximation centered around v = 0. All derivatives of $L_v(L_0', L_0'' \dots)$ are matricies of constant coefficients, in the same way that for a Taylor expansion, $f(0), f'(0), \dots$" are constants.

$$L_v = I + L_0'v + O(v^2)$$

Where I is the 2x2 identity matrix and $O(v^2)$ is a 2x2 matrix whose elements are all functions of v^2 and higher order terms which we do not (yet) care about.

$$O(v^2) = \begin{pmatrix} o(v^2) & o(v^2) \\ o(v^2) & o(v^2) \end{pmatrix}$$

Now, it is finally some time for you to do math. If we let:

$$L_0' = \begin{pmatrix} a & b \\ d & e \end{pmatrix}$$

Give me the 2x1 matrix for:

$$L_v \begin{pmatrix} t \\ x \end{pmatrix}$$

In terms of $a, b, d, e, v, o(v^2), t, x$ (check in with one of the captains after you get the matrix). Group your expressions by t and x.

Now, we must turn towards solving for the coefficients. In order to do so, I am going to define a new operation: reflection. And all this does is replace x with -x; if we describe an event E in frame F as $F(E) = (t, x)$, we describe it in frame $\bar{F}$ as $\overline{F(E)} = (t, -x)$. Use this reflection transformation, in addition to how it relates to velocity, to solve for a and e by plugging this into the matrix you found in the last part and looking at like terms (check in the captains after you're done).

Say that a particle is moving at velocity v in frame F. Let's construct a frame F' which is also moving at velocity v relative to F. Recall that we normalized all frames such that event O occurs at (0,0) in both F and F'; therefore, in frame F', the x coordinate of the particle is always 0, and the particle is traveling along the (t', 0) line. In frame F, x = vt for all t, so we can re-write F(t, x) as F(t, vt).

To put that all together:

$$\begin{pmatrix} t' \\ 0 \end{pmatrix} = L_v \begin{pmatrix} t \\ vt \end{pmatrix}$$

Now, solve for d by again equating like terms and check in with the captains.

Finally, let us turn our attention towards b. To do so, let's consider the motion of light. Per the postulates of relativity, the speed of light is constant in all frames of reference, which means that for photons, $c = \frac{x}{t} = \frac{x'}{t'}$. Rearranging, $x = ct$ and $x' = ct'$, so we have $L_v(t, ct) = (t', ct')$. Use this information to solve for b.

And now write out the Lorentz transform of the first order. Compare this to what you already know about transformations between reference frames (the x component should look very familiar).

But wait, there's more! Let's find the Lorentz transform to the second order now. But, before you stone me to death, it's just one hop this time. Cha-cha real smooth.

In the second order approximation:

$$L_v = I + L_0'v + \frac{1}{2}L_0''v^2 + O(v^3)$$

If we shift from a rest frame to a frame moving at v and then shift again to a frame moving at -v, we are back where we started. Symbolically:

$$L_{-v} \circ L_v = I$$

Plug the second order approximation into the equation above, and solve for L_0'' in terms of L_0'. Hint: only foil what you gotta foil for the second order approximation - we can ignore anything v^3 and higher. Then, write out:

$$L_v \begin{pmatrix} t \\ x \end{pmatrix}$$

and check in with the captains again.

We could keep going an eventually uncover a pattern, but in the interest of time, take the Taylor expansion of $\frac{1}{\sqrt{(1-\frac{v^2}{c^2})}}$. With that in mind, and assuming the pattern continues for higher order terms, write out the full, infinite order, Lorentz transform in terms of t, v, x, c, and $\frac{1}{\sqrt{(1-\frac{v^2}{c^2})}}$.

And there we go. That's the Lorentz transform.

Finally, let us relate this formulation of the Lorentz transform back to the space-time coordinate system that we talked about earlier - namely, (ct, x). We want to find a matrix A such that:

$$A \begin{pmatrix} ct \\ x \end{pmatrix} = \begin{pmatrix} ct' \\ x' \end{pmatrix}$$

To simplify notation, let $\gamma = \frac{1}{\sqrt{(1-\frac{v^2}{c^2})}}$ and $\beta = \frac{v}{c}$. Solve for A. Quick check: this matrix should be invertible. Per linear algebra, a quick check for invertibility is a check for a non-zero determinant. Do that check real quick.

Remember how we talked about coordinates "shearing" and "rotating"? Let's look at the rotational aspect of that. If you recall, the standard Cartesian rotation matrix is as follows:

$$\begin{pmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{pmatrix}$$

With that in mind, let us introduce the concept of rapidities.

$$tanh(\theta_v) = \frac{v}{c}$$

Express the Lortentz transform matrix (the one which operates on (ct, x)) in terms of hyperbolic cosines and sines.

Next, we will look specifically how time, distance, and velocities transform under the Lorentz transform.

Suppose Sambuddha is running (it does happen from time to time, believe it or not) at a constant velocity v and you are standing on the ground staring in awe. Letting x = vt (and thereby x' = 0), solve for t in terms of t' and interpret amongst yourselves.

Now, suppose that Sambuddha whips out a meter stick, and that you and him both measure the length of the stick simultaneously; that is to say, ct = ct' = 0. Solve for x in terms of x', and interpret.

Now, since Sambuddha has pockets the size of Tartarus, he whips out a Django Unchained style revolver. Scratch that, he pulls out two and puts on a burgundy suit. He fires the bullet at a velocity u'. You measure the bullet to be going a speed u. Let u' = $\frac{dx'}{dt'}$ and u = $\frac{dx}{dt}$, and solve for u. This is called relativistic velocity addition.

One final thing - let's bring rapidities up in this joint.

Let $tanh(\theta_v) = \frac{v}{c}$, $tanh(\theta_u) = \frac{u}{c}$, and $tanh(\theta_{u'}) = \frac{u'}{c}$. Using the fact that $tanh(\theta_1+\theta_2) = \frac{tanh(\theta_1)+tanh(\theta_2)}{1+tanh(\theta_1)tanh(\theta_2)}$, express θ_u in terms of θ_u' and θ_v.

3 4 Vectors

Let us go back to the position 4-vector $X = (ct, x, y, z)$.

Prove that $(ct')^2 - x'^2 = (ct)^2 - x^2$ is constant. This is an example of an "invariant": while observers may disagree on time elapsed and length, they will always agree on the "length" of the position 4-vector.

Compute $\frac{dX}{d\tau}$, where τ is the proper time. Then multiply the ensuing velocity vector by the rest mass m_0 to get the particle's momentum. Think about what both the position component of the vector means.

Finally, consider the temporal component of the momentum vector (let's call this p_0). Expand it out with a Taylor approximation, and then multiply both sides by c. What do you notice about the second term of the expansion? What does this tell you about the first term? So, what is cp_0 a measure of?

That's right fools. We have $E = \gamma m_0 c^2$. To close things out, prove that $E^2 = (mc^2)^2 + (pc)^2$, where $p = m_0 v$.