Projectile Motion Lecture

M. Yu

September 2015

1 Introduction

So you've probably heard a lot about projectile motion with no air resistance, so we will demonstrate how we can use calculus to do air resistance. Projectile motion problems are also have much more depth and complexity than expected. We assume that you already know the big 5 discussed in the beginning of any physics class so we will not review them. We will attempt a hard problem of finding the lunch angle to maximize the flight path of a projectile. Notice, not the range, the actual path in the air.

2 Air Resistance

Remember, air resistance is a resistive force. Notice I said force, this means that you should probably be using Newton's second law. Lets imagine we have a ball falling from a height h and the force of drag $F_d = -kv$. If a ball is falling, the force of drag should point opposite the direction of the ball, this is the reason for the negative. The force equation looks like this:

$$F = -mg - kv \tag{1}$$

Lets do a little math trick and let $k = m\beta$, no problems here, you can aways find m and β that exist.

$$\begin{gathered} m\frac{dv}{dt} = -mg - m\beta v \\ \int_0^{v(t)} \frac{dv}{g + \beta v} = -\int_0^t dt \end{gathered} \tag{2}$$

Integrate (2) letting the denominator of the LHS be u and remember to change the bounds of integration. You should get $\ln(1 + \beta \frac{v(t)}{g}) = -\beta t$. Solving for $v(t)$ we get

$$v(t) = -\frac{g}{\beta}\left(1 - e^{-\beta t}\right). \tag{3}$$

We can do the same thing as we did before by letting $\frac{dy}{dt} = v(t)$, separating variables, and integrating to give position.

$$\begin{gathered} \int_h^{y(t)} dy = -\frac{g}{\beta}\int_0^t \left(1 - e^{-\beta t}\right) dt \\ y(t) = h - \frac{g}{\beta}\left(t - \frac{1}{\beta}\left(1 - e^{-\beta t}\right)\right) \end{gathered} \tag{4}$$

3 Maximum Flight Path

At what angle do we launch with some initial velocity so that we get the maximum flight path. You can solve the maximum range (x distance) quite easily with some trig and algebra, but this problem will take some calculus. We will assume that the projectile takes some path, call that path $y(t)$. Consider the time which the particle is in the air, we can solve that easily to be $\frac{2v \sin\theta}{g}$.

If you recall from calculus, we can write the equation for the path length to be

$$\begin{aligned} P &= \int_0^{\frac{2v\sin\theta}{g}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\ &= 2\int_0^{\frac{v\sin\theta}{g}} \sqrt{(v\cos\theta)^2 + (v\sin\theta - gt)^2} dt \end{aligned} . \tag{5}$$

The last line of (5) stems from the fact that we are splitting the path up in half, going up, and then multiplying by 2 to get the going down part. Symmetry allows us to do this. To tackle this beast, we can do the integral, then take a derivative with respect to theta. It is easy to see that the equation for P will indeed be in terms of theta, so have fun doing a ton of algebra. Wofiram is of course allowed as the final answer will have to be numerically done. It turns out $\theta \approx 56.5$.