Differential Equations - Oscillations

C. Liu

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1 Introduction

Definition 1. *Differential Equation: An equation involving an unknown function and at least one of its derivatives*

Because physics often seeks to find the relationship between different physical properties as some variable changes continuously, solving differential equations is an integral (pun intended) part of physics. For instance, take:

$$\frac{\partial u}{\partial t} = h^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

This is the heat equation, and a partial differential equation. As the name implies, partial differential equations involve partial derivatives. Luckily, we won't have to handle those in oscillations.

In our example of a oscillator today, we'll work with springs. Note that these are obviously not the only possible oscillators; specifically, we are skipping pendulums due to lecture time constraints. The differential equation for the position of a spring as a function of time is:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = f(t)$$

Where $x(t)$ is position as a function of time, b is a linear drag coefficient, k is the spring constant, and $f(t)$ is an arbitrary "driving" function, or an external force, on the spring's movement. If $b = 0$ and $f(t) = 0$, then this spring would be simple harmonic oscillator. We will define both an oscillator and a simple harmonic oscillator after we solve the damped spring problem.

2 Solving the Damped Spring Problem

First let's work with a spring with no external force. Notice that both b and k are constants. Therefore, we know that $x(t)$ is a function whose derivatives are constant multiples of the function itself. A function that fulfills that requirement is $x(t) = e^{rt}$ for some constant r. Plugging this into our differential equation, we get:

$$mr^2e^{rt} + bre^{rt} + ke^{rt} = 0$$

$$e^{rt}(mr^2 + br + k) = 0$$

$$mr^2 + br + k = 0$$

Solving this quadratic we get that $r = -\frac{b}{2m} \pm \frac{\sqrt{b^2 - 4mk}}{2m}$. Letting $\gamma = \frac{b}{2m}$, we can also write the two possible values of r as $-\gamma \pm \sqrt{\gamma^2 - \frac{k}{m}}$. These solutions let us see that there are 3 cases determined by the sign of the discriminant (positive, 0, or negative) in solving for $x(t)$. We will solve the case for which the discriminant is negative. The other two cases are left as exercises for the members of physics team.

When the discriminant is negative, $r = -\gamma \pm i\sqrt{\frac{k}{m} - \gamma^2}$. The 2 associated equations $x(t)$ are

$$x_1(t) = e^{-t\gamma} \left(\cos\left(t\sqrt{\frac{k}{m} - \gamma^2} \right) + i \sin\left(t\sqrt{\frac{k}{m} - \gamma^2} \right) \right)$$

and

$$x_2(t) = e^{-t\gamma}\left(\cos\left(t\sqrt{\frac{k}{m} - \gamma^2}\right) - i\sin\left(t\sqrt{\frac{k}{m} - \gamma^2}\right)\right)$$

However, we are describing a real system. As a result, the imaginary numbers are unwelcome and we would like to get rid of them. Here, we introduce an important theorem in differential equations:

Theorem 1. *Principle of Superposition: If $x_1, x_2, \ldots, x_n$ are solutions of a homogeneous and linear differential equation on some interval I, then*

$$x(t) = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

(where the c_i are constants) is also a solution to the differential equation on I.

The Principle of Superposition applies both to real and complex constants, as you will prove or have proven in Analysis 2B. Because we care only about the real solutions $x(t)$ to our equation for a damped spring with no external forces, we choose the c_1 and c_2 for which $c_1x_1 + c_2x_2 = x(t)$ is real. Ultimately, we get:

$$x(t) = e^{-t\gamma}\left(D\cos\left(t\sqrt{\frac{k}{m} - \gamma^2}\right) + N\sin\left(t\sqrt{\frac{k}{m} - \gamma^2}\right)\right)$$

(Note that I'm running out of letters to use as variables.) $D, N \in \mathbb{R}$. Because the linear combination of sinusoids is a sinusoid, we may also write $x(t)$ as:

$$x(t) = Ae^{-t\gamma}\cos(w_dt + \phi)$$

Where $Ae^{-t\gamma}$ and $-Ae^{-t\gamma}$ are "envelope functions" between which $x(t)$ must vary, $w_d = \sqrt{\frac{k}{m} - \gamma^2}$ is the damped frequency of the spring's oscillation, and ϕ is the phase shift angle fulfilling $\cos(\phi) = \frac{D}{\sqrt{D^2+N^2}}$ and $\sin(\phi) = \frac{-N}{\sqrt{D^2+N^2}}$.

Now we can finally define general and simple harmonic oscillators:

Definition 2. *Oscillator: A physical system that experiences a restoring force towards equilibrium, usually proportional to the distance from equilibrium.*

Definition 3. *Simple Harmonic Oscillator: A physical system of undergoes simple harmonic motion, or sinusoidal motion about an equilibrium point with a constant amplitude and frequency.*

3 Considerations

Now that the fundamental case for springs is solved, there are some interesting variations of this problem that you should consider. This will help you with both the problem set and developing some physics intuition in general.

1. Matthew is sitting on a swing. Patrick is pushing Matthew on the swing. What happens as Patrick varies when and how often he pushes Matthew on the swing?

2. Consider a cart with a spring inside. The spring and is given some initial displacement $\delta(x)$. The cart begins travelling at a constant velocity v across the ground, but the ground's shape is sinusoidal. In other words, the position of the cart as it moves is given by $x_g(t) = A\sin(\omega_gt)$ for some real A and ω_g. What happens to $x_s(t)$, the position function of the spring? This works for both damped and undamped springs.

3. Recall that springs are a very specific instance of oscillations. Oscillatory motion, even simple harmonic motion, occurs in many places. What other instances can you think of? Be very creative.